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An Improvement to the Method of Integral Relations

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An improvement to the Method of Integral Relations (MIR) using orthonormal weighting functions is described. The method is applied to two test problems in boundary layer flow for which similarity solutions exist. The improved method is written for arbitrary order and it is shown to give close agreement with the similarity solutions as the order is increased.

The formulation of the Method of Integral Relations (MIR) for treatment of boundary layer flows was first presented by Dorodnitsyn [1] in 1960. Since then it has been applied to a variety of problems dealing with both attached and separated flows [2–7]. Boundary layer flows are relatively simple to treat due to the parabolic nature of the underlying governing partial differential equations. The MIR has also been applied to elliptic and mixed elliptic–hyperbolic problems; for example, Belotserkovskii [8] applied the MIR to the supersonic blunt body problem. However, the nature of the governing equations in such cases requires that, within the framework of the MIR, the problem be treated as a two-point boundary value problem. Due to the manner in which Belotserkovskii set up the MIR formulation the downstream boundary condition appeared as a saddle point where the flow becomes sonic.

In general, application of the MIR to elliptic or mixed type problems will require an additional iterative technique in order to link conditions at the "downstream" boundary with conditions at the "upstream" boundary (i.e., point at which integration of the ordinary differential equations begins). The multiple shooting method [9] is well suited to this problem. Meng [10] has used the multiple shooting method in conjunction with the GTT (or Telenin's) method to consider the Reentry Base Heating problem. The GTT method is similar to the MIR in that it modifies the governing partial differential equations so that they may be treated as ordinary differential equations.

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The MIR is based on the representation of the flow variation in one or more directions analytically, and, in principle, such variation can be exactly accounted for by including a sufficiently large number of terms, N, in the analytic expressions. In practice the maximum value typically considered is $N = 4$ or 5. For the elementary case of two-dimensional similarity solutions the equations reduce to algebraic equations and solutions can readily be obtained up to larger values of N (typically $N = 10$).

There are two reasons for normally restricting N to 4 or 5. These reasons can be illustrated by considering the case of a two-dimensional incompressible boundary layer flow parallel to a wall. After suitable transformation the basic integral equation becomes (see [1])

$$\frac{d}{d\xi}\int_0^1 \theta u f(u)\,du = \frac{\dot{V}}{V}\int_0^1 \theta f'(u)(1-u^2)\,du - [f'(u)\,\tau]_{\text{wall}} - \int_0^1 f''(u)\,\tau\,du. \quad (1)$$

Here ξ is the coordinate parallel to the wall, η is the transverse coordinate, u is the ξ component of velocity, $\tau = \partial u/\partial\eta$, $\theta = 1/\tau$, $\dot{V}$ is the external velocity gradient, and f is the weighting function. In his original presentation Dorodnitsyn chose the linearly independent set of functions,

$$f_k = (1-u)^k, \quad (2)$$

and the following representation for the reciprocal of the normal velocity gradient,

$$\theta = \frac{1}{1-u}\left[a_0 + \sum_{j=1}^{N-1} a_j u^j\right]. \quad (3)$$

To evaluate the N unknown coefficients a_j, Eq. (1) must be evaluated with N different weighting functions, f_k, $k = 1,\ldots, N$, given by Eq. (2). For N large the difference between f_N and f_{N-1} is very small and the corresponding evaluations of Eq. (1) are almost linearly dependent. The result of applying f_k to Eq. (1) N times can be expressed in matrix notation as

$$[B][\partial a_j/\partial\xi] = [C] \quad (4a)$$

and

$$[\partial a_j/\partial\xi] = [B]^{-1}[C]. \quad (4b)$$

The matrices $[B]$ and $[B]^{-1}$ are progressively more ill-conditioned as N increases.

The second difficulty associated with N becoming large is the algebraic labor required in the actual calculation of matrices $[C]$ and $[B]$. For incompressible flow past adiabatic walls requiring solution of Eq. (1) only, this is not particularly excessive, but for more complicated problems in compressible, three-dimensional or separated flow Eq. (1) will be replaced by a system of coupled nonlinear integrodifferential equations. For such equations the effort required to obtain analytic expressions for [B] and [C] for large N would be prohibitive.

In the present method both these difficulties are overcome. Two test cases will be considered to illustrate the method. The first concerns the attached boundary layer flow of an incompressible fluid over a wedge for various included angles. For such a flow a similarity solution due to Falkner and Skan [11] is available. For this problem comparisons are made with the conventional application of MIR [1]. The second test case deals with supersonic boundary layer flow on a cone.

The basis of the present method is to take Dorodnitsyn's weighting functions and to generate from them a set of orthonormal functions, g_j. The functions g_j are used to replace the weighting function f_j and also to replace powers of u in the analytic representation for θ. The result of evaluating Eq. (1) then permits Eq. (4b) to be replaced by

$$\partial a_j / \partial \xi = C'(j). \tag{5}$$

In contrast to the original formulation no matrix inversion is required. Also, the problem of tedious algebraic calculations is avoided by evaluating $C'(j)$ numerically.

The functions g_j have the form

$$g_j = \sum_{k=1}^{j} b_{kj} \cdot f_k , \tag{6}$$

where f_k are the Dorodnitsyn weighting functions defined by Eq. (2) and the coefficients b_{kj} are evaluated using the Gram–Schmidt orthonormalization process [12]. For this problem, the values of the leading coefficients are $b_{11} = 6^{1/2}$, $b_{12} = -(30)^{1/2}$, $b_{22} = (120)^{1/2}$, etc. A definition of the inner product with respect to a given weighting function, w, is required. This is

$$(g_i , g_j) = \int_a^b g_i(x)\, g_j(x)\, w(x)\, dx. \tag{7}$$

Then $\{g_i(x)\}$ is a set of orthonormal functions if

$$\begin{aligned} (g_i , g_j) &= 1, \qquad i = j; \\ &= 0, \qquad i \neq j. \end{aligned}$$

To apply this to the first model problem Eq. (3) is replaced by the following representation for θ,

$$\theta = \frac{1}{1-u} \left[b_0 + \sum_{j=1}^{N-1} b_j g_j \right]. \tag{8}$$

The nonorthogonal leading term, b_0, is retained to ensure the correct behavior for θ at the outer edge of the boundary layer. Replacing f in Eq. (1) by g and expanding the right-hand side of Eq. (1) leads to

$$\frac{d}{d\xi}\int_0^1 \left[b_0 + \sum_{j=1}^{N-1} b_j g_j\right] g_k \frac{u}{1-u}\, du = C'(k), \qquad k = 1,\ldots, N. \tag{9}$$

Comparison of Eqs. (7) and (9) shows that the appropriate orthonormal weighting function is $w(u) = u/(1-u)$.

Previous applications of MIR have all used the same weighting function, $(1-u)^k$, as suggested by Dorodnitsyn [1]. However, in switching to the corresponding orthonormal functions the weighting function, w, may vary from problem to problem. Thus for the inclined cone problem [5] the correct weighting function is $w(u) = 1/(1-u)$. Clearly different weighting functions combined with the same basic functions, $(1-u)^k$, lead to different orthonormal functions.

With g_k appropriately chosen Eq. (9) becomes

$$\frac{db_0}{d\xi}\int_0^1 g_k w\, du + \frac{db_k}{d\xi} = C'(k), \qquad k = 1,\ldots, N-1. \tag{10}$$

When $k = N$ Eq. (9) reduces to

$$\frac{db_0}{d\xi}\int_0^1 g_N w\, du = C'(N). \tag{11}$$

From Eqs. (10) and (11) it follows that

$$\frac{db_k}{d\xi} = C'(k) - C'(N)\left[\int_0^1 g_k w\, du\right]\Big/\left[\int_0^1 g_N w\, du\right]. \tag{12}$$

Consequently the $db_k/d\xi$ are given explicitly and an ill-conditioned matrix is avoided. Integrals of the form $[\int_0^1 g_k w\, du]$ can be evaluated once and for all.

$$C'(k) = \frac{\dot{V}}{V}\int_0^1 \theta g_k'(u)(1-u^2)\, du - [g_k'(u)\,\tau]_{\text{wall}} - \int_0^1 g_k''(u)\,\tau\, du. \tag{13}$$

The integrals on the right-hand side of this equation have been evaluated numerically using a composite Simpson's rule and m evaluations of the integrands at equal intervals in the range $u = 0$ to 1. Typically for $m \geqslant 30$ $C'(k)$ is accurate to six decimal places.

To compare with the conventional application of MIR Eqs. (12) and (13) have been integrated numerically using a fourth-order Runge–Kutta scheme. Starting data for the integration are provided by the Falkner–Skan solutions at $\xi = 1$.

Three cases have been considered:

(a) $\beta = 0.5$ (favorable pressure gradient),
(b) $\beta = 0$ (flat plate),
(c) $\beta = -0.14$ (unfavorable pressure gradient),

where

$$\beta = (2\xi)^{1/2}\, \dot{V}/V.$$

Solutions for these cases by the conventional application of MIR could be found by reducing Eq. (1) to a set of algebraic equations [1]. However, to present a situation which is representative of more general nonsimilar flow the integration of the equations given by Dorodnitsyn [1] (corresponding to Eq. (1)) has been carried out numerically.

Figures 1, 2, and 3 show the percentage difference between the approximate and exact wall shear stress for various values of β. For favorable and zero pressure gradient the orthonormal functions give more accurate results at the same order. This does not seem to be the case for the unfavorable pressure gradient shown here. However, the main point of this example is to show that higher orders using the orthonormal representation approach more closely the exact solution, and that the extension to higher order is trivial compared with the labor and the ill-conditioned results associated with traditional MIR.

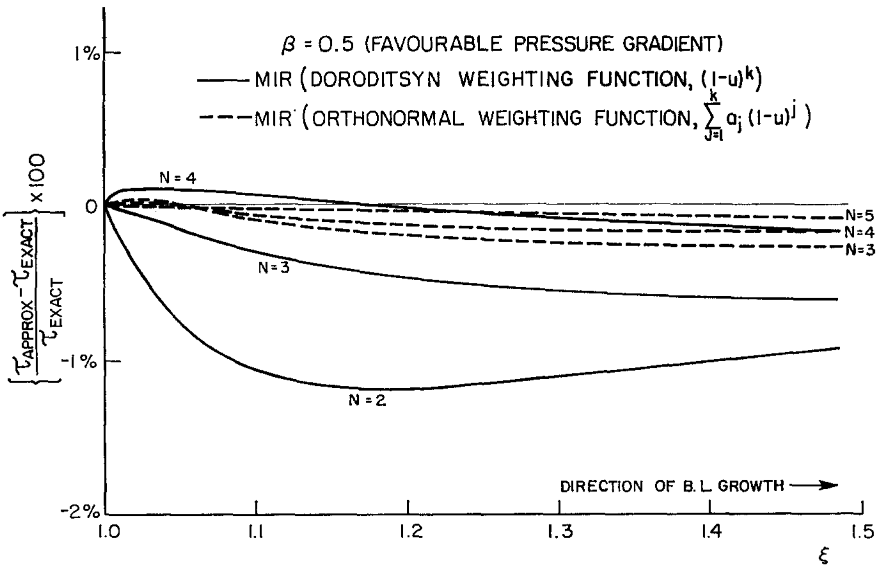

FIG. 1. Comparison of orthonormal MIR and conventional MIR for Falkner–Skan solutions.

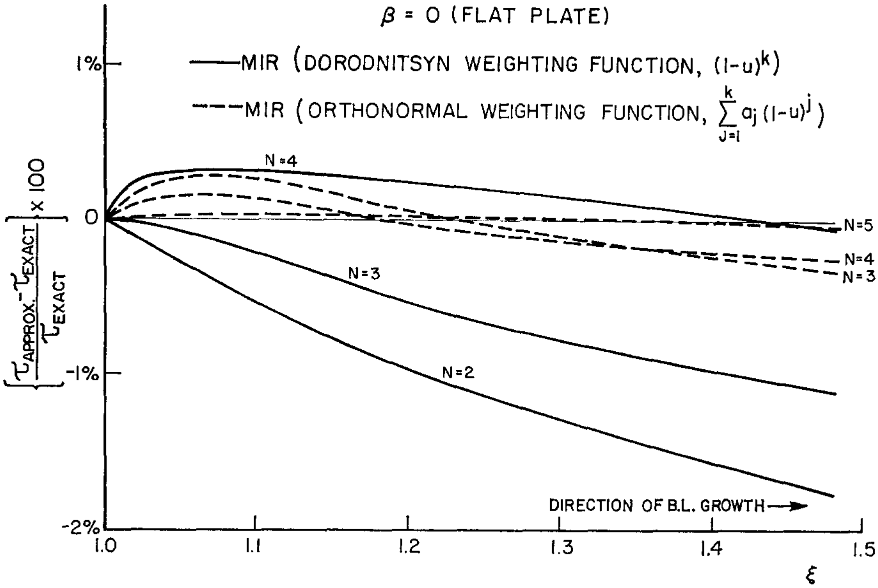

FIG. 2. $\beta = 0$ (flat plate).

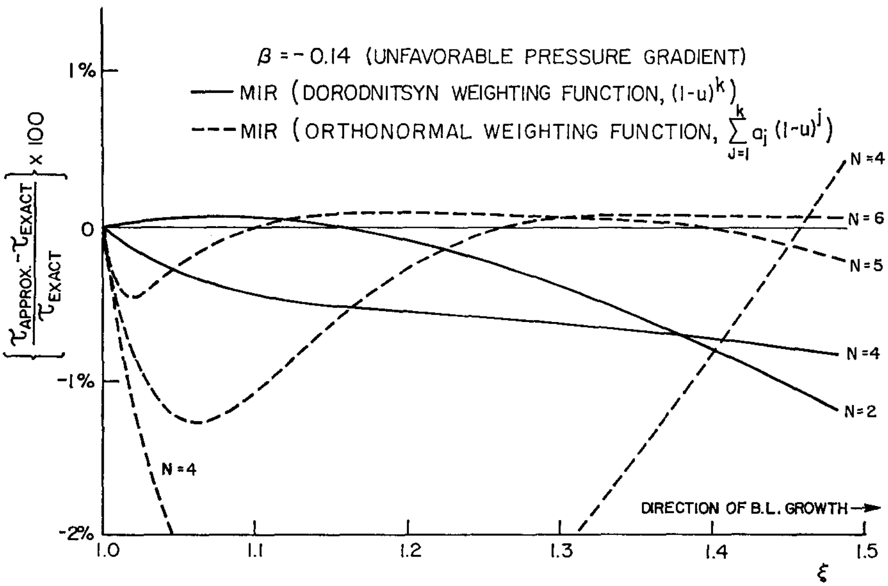

FIG. 3. $\beta = -0.14$ (unfavorable pressure gradient).

The second test case, that of compressible boundary layer flow over a cone at zero angle of attack, is more complex because it involves both variable density and heat transfer. For this case the exact integral relations, equivalent to Eq. (1), are

$$\int_0^1 g_j(u) \frac{u}{\tau}\, du = b_1 \left\{ g_j'(0)\, \tau_{\text{wall}} + \int_0^1 g_j''(u)\, \tau\, du \right\}, \tag{14}$$

$$\begin{aligned}\int_0^1 g_j(u) \frac{su}{\tau}\, du = {} & b_1 g_j'(0)\, s_{\text{wall}} \tau_{\text{wall}} + b_2 g_j(0) \left.\frac{\partial s}{\partial u}\right|_{\text{wall}} \tau_{\text{wall}} \\ & + (b_1 + b_2) \int_0^1 g_j'(u)\, \tau \frac{\partial s}{\partial u}\, du + b_1 \int_0^1 g_j''(u)\, \tau s\, du \\ & + b_3 \int_0^1 g_j''(u)\, \tau u\, du. \end{aligned} \tag{15}$$

Here τ is a modified normal gradient of the velocity along the cone generator (u), s is a nondimensional total enthalpy, g_j is the orthonormal weighting function, and b_1, b_2, b_3, and b_4 are parameters that depend on the flow conditions outside the boundary layer, the free stream conditions, and the cone geometry. Equations (14) and (15) have been obtained from the three-dimensional compressible boundary layer equations applied to the more general case of a cone at angle of attack. The original equations were, in succession, subjected to Howarth, Mangler, Blasius, and Crocco transformations. Finally, the limit of zero angle of attack was applied.

To solve these equations the following representations are assumed for the terms u/τ and su/τ.

$$\frac{u}{\tau} = \frac{1}{(1-u)} \left[b_{01} + \sum_{j=1}^{N-1} b_{j1} g_j(u) \right], \tag{16}$$

$$\frac{su}{\tau} = \frac{1}{(1-u)} \left[b_{02} + \sum_{j=1}^{N-1} b_{j2} g_j(u) \right]. \tag{17}$$

Certain of the coefficients b_{ji} are obtained by satisfying the boundary conditions. At the outer edge of the boundary layer $s = 0$ and $g_j(1) = 0$ for all j. From this it follows that $b_{02} = 0$. At $u = 0$, $s = s_w$, which has to be given *a priori*. This leads to

$$b_{01} + \sum_{j=1}^{N-1} b_{j1} g_j(0) = 0, \tag{18}$$

$$\sum_{j=1}^{N-1} b_{j2} g_j(0) = 0, \tag{19}$$

and

$$s_w = \left[\sum_{j=1}^{N-1} b_{j2} g_j'(0)\right] \Big/ \left[\sum_{j=1}^{N-1} b_{j1} g_j'(0)\right]. \tag{20}$$

To obtain the other $N-2$ coefficients, Eqs. (14) and (15) are applied $N-2$ times with different weighting functions $g_j(u)$, $j = 1,..., N-2$. As in the first test case, the right-hand sides of Eqs. (14) and (15) are evaluated numerically using a composite Simpson's rule and m evaluations of the integrand.

It may be noted that Eqs. (14) and (15), with the introduction of Eqs. (16) and (17), can be treated as algebraic relations. The basic problem is then to determine values of the coefficients b_{ji} that will satisfy Eqs. (14) and (15). To do this use has been made of an iterative technique based on a function minimization method due to Powell [13]. Making use of the orthonormal property, Eqs. (14) and (15) can be written

$$E_j = b_{01} \int_0^1 \frac{g_j}{(1-u)}\, du + b_{j1} - A_j = 0, \tag{21}$$

$$F_j = b_{j2} - B_j = 0, \tag{22}$$

where A_j is the right-hand side of Eq. (14) (evaluated numerically) and B_j is the right-hand side of Eq. (15) (evaluated numerically). For an arbitrary choice of the b_{ji}'s, E_j and F_j will be nonzero. Powell's method modifies the current values of b_{ji} until

$$G = \sum_{j=1}^{N-2} (E_j^2 + F_j^2)$$

is a minimum. Only solutions for which $G_{\min} = 0$ are meaningful.

Since the method of solution is iterative, starting values for the b_{ji}'s are required. Once $G_{\min} = 0$ for a particular value of N has been obtained, the converged values of b_{ji} are used as starting values for the solution of order $N+1$. The two extra coefficients, $b_{N+1,1}$ and $b_{N+1,2}$, are initially set equal to zero. Thus, at the beginning of the iteration to find the solution of order $N+1$, only the residuals E_{N-1} and F_{N-1} will contribute to G. This procedure effectively reduces the starting value problem to finding the starting values of b_{ji} for the lowest-order solution sought. Since four of the b_{ji}'s are determined by the boundary conditions the lowest-order solution that can satisfy Eqs. (14) and (15) is for $N = 3$. In this case the two free b_{ji}'s are obtained by guessing suitable values for u/τ and su/τ at $u = 0.5$ and using Eqs. (16) and (17) to give the corresponding b_{ji}'s.

The above technique could contribute to the solution of an elliptic or mixed type problem where multiple shooting is employed. Thus a solution would be sought for the elliptic problem using the lowest order of the MIR that is feasible.

This would minimize the number of unknowns that require iterative solution using the multiple shooting method. The solution for the lowest-order MIR would then form the first estimate for the MIR solution of the next-higher order, as indicated above.

In the limit as N and m tend to infinity it is expected that the approximate solution represented by Eqs. (16) and (17) will approach the exact solution of Eqs. (14) and (15). To check this, solutions have been obtained up to $N = 14$ and $m = 128$. m is the number of points used to span the boundary layer when evaluating A_j and B_j. The results for the nondimensional skin friction parameter, $Re_x^{1/2}\, C_f$, are shown in Table I. Re_x is a Reynolds number based on the local

TABLE I

Variation of $Re_x^{1/2}\, C_f$ with N and m

N	$m = 24$	$m = 40$	$m = 64$	$m = 96$	$m = 128$	$m = \infty$
3	1.28452	1.28767	1.28818	1.28955	1.28996	
4	1.15341	1.15570	1.15698	1.15768	1.15803	
5	1.14788	1.14966	1.15064	1.15118	1.15145	
6	1.14768	1.14934	1.15025	1.15074	1.15099	
7	1.14724	1.14879	1.14961	1.15006	1.15028	1.15050
8	1.14716	1.14868	1.14947	1.14989	1.15010	1.15031
9	1.14711	1.14862	1.14939	1.14979	1.14999	1.15019
10	1.14708	1.14859	1.14935	1.14975	1.14994	1.15013
11	1.14705	1.14857	1.14933	1.14972	1.14990	1.15008
12		1.14856	1.14932	1.14970	1.14988	1.15006
13		1.14855	1.14931	1.14969	1.14987	1.15005
14		1.14859	1.14930	1.14969	1.14987	1.15005
∞			1.14929	1.14969	1.15987	1.15005

external flow conditions. The skin friction parameter is directly proportional to the value of τ at the wall. It can be seen that the tabulated values of $Re_x^{1/2}\, C_f$ vary smoothly enough to permit extrapolation to $N = \infty$ and $m = \infty$ as long as the entries corresponding to a small m coupled with a large N are discounted (e.g., $m = 40$, $N = 14$). Clearly this is not an efficient combination in any case.

The test case of a cone at zero angle of attack was deliberately chosen because it possesses a similarity solution. The value of $Re_x^{1/2}\, C_f$ from the similarity solution is 1.15028. The extrapolated value from Table I, i.e., $Re_x^{1/2}\, C_f = 1.15005$, compares favorably with the similarity solution.

A more stringent test is obtained if the heat transfer to the wall is considered. The variation of the nondimensional heat transfer parameter, $Pr\, Re_x^{1/2}\, St$ is shown in Table II. Pr is the Prandtl number, St is the Stanton number, and the parameter $Pr\, Re_x^{1/2}\, St$ is proportional to the gradient $\partial s/\partial u$ at the wall.

TABLE II

Variation of $Pr\ Re_x^{1/2}\ St$ with N and m

N	$m = 24$	$m = 40$	$m = 64$	$m = 96$	$m = 128$
3	0.29255	0.29038	0.29131	0.28992	0.28970
4	0.47669	0.47652	0.47663	0.47660	0.47657
5	0.48676	0.48659	0.48652	0.48643	0.48646
6	0.48855	0.48825	0.48813	0.48807	0.48804
7	0.49553	0.49472	0.49454	0.49448	0.49446
8	0.49832	0.49705	0.49675	0.49667	0.49664
9	0.50546	0.49947	0.49865	0.49848	0.49843
10	0.49455	0.50058	0.49994	0.49975	0.49970
11	0.61802	0.50629	0.50147	0.50080	0.50067
12		0.49778	0.50165	0.50146	0.50137
13		0.55076	0.50552	0.50258	0.50212
14		0.24730	0.50039	0.50238	0.50240

Table II indicates that the heat transfer parameter is not as well behaved as $Re_x^{1/2}\ C_f$ with increasing N. An unrealistic oscillation with increasing N develops if m is small. From these results and others not shown it is necessary to increase m as N increases. For the range of N and m shown, it appears that only the values corresponding to $m = 128$ are free from oscillation; consequently, no attempt has been made to extrapolate the results. For this case an accurate solution, based on similarity solutions, is given by Hantzsche and Wendt [14] and by Young [15]. For a Prandtl number of 0.7 this solution is $Pr\ Re_x^{1/2}\ St = 0.50644$. The tabulated value at $N = 14$ and $m = 128$ is within 1% of Young's value for $Pr\ Re_x^{1/2}\ St$. It is noticeable that a fairly large value of N is required before the heat transfer parameter becomes close to Young's solution. The relatively poor behavior of the heat transfer parameter is not altogether unexpected when it is recalled that $\partial s/\partial u$ at the wall depends on differentiating Eqs. (16) and (17) twice. However, for the more general case of an inclined cone at large angle of attack the computed heat transfer parameter shows good agreement with experimental results.

In conclusion, an improvement to Dorodnitsyn's formulation of MIR is proposed. The main advantage of the present formulation is that it can readily be extended to higher orders.

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